# Proof of the cosmic no-hair conjecture for quadratic homogeneous cosmologies

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**Abstract.** We prove the cosmic no-hair conjecture for all orthogonal Bianchi cosmologies with matter in the $R + \beta R^2$ theory using the conformally equivalent Einstein field equations, with the scalar field having the full self–interacting potential, in the presence of the conformally related matter fields. We show, in particular, that the Bianchi IX universe asymptotically approaches de Sitter space provided that initially the scalar three–curvature does not exceed the potential of the scalar field associated with the conformal transformation. Our proof relies on rigorous estimates of the possible bounds of the so-called Moss-Sahni function which obeys certain differential inequalities and a non-trivial argument which connects the behaviour of that function to evolution of the spatial part of the scalar curvature.

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## 1. Introduction

Attempts to tackle to isotropization problem in relativistic cosmology date at least since the pioneering work of Collins and Hawking [1] who proved isotropization theorems for certain classes of orthogonal Bianchi spacetimes. A further advance is the recent work by Heusler [2] which extended the Collins–Hawking results to the case of a minimally coupled scalar field matter source with a general convex potential. Interest in a particular approach to the isotropization problem in cosmology renewed after the advent of inflation as a mechanism for solving the problem and focused on proving the so–called cosmic no–hair conjecture. This conjecture roughly speaking states that general cosmological initial data sets when evolved through the gravitational field equations are attracted (in a sense that can be made precise) by the de Sitter space of inflation. In other words, if this conjecture is true, inflation is a 'transient' attractor of such sets, these in turn quickly isotropize after the inflationary period and the inflationary regime could thus be regarded as 'natural' in view of its *prediction* of the (observed) large–scale homogeneity and isotropy of the universe.

Higher derivative quantum corrections to the gravitational action of classical general relativity are generally expected to play a significant role at very high energies where a quantum gravitational field will presumably dominate. It is not unreasonable to consider classical cosmology in theories coming out of such nonlinear (higher derivative) gravitational Lagrangians and in fact, one expects that there exist close links between properties of such "higher derivative cosmologies" and those of general relativistic cosmology. It is obvious that the resolution of aspects of the singularity, isotropization and recollapse problems of cosmological spacetimes is of paramount importance also in this extended framework.

In this paper, we prove the cosmic no–hair conjecture for all Bianchi (including type IX) matter cosmologies which are derived from a quadratic action in the scalar curvature that is of the form $L = R + \beta R^2 + L_m$ where $L_m$ denotes the matter Lagrangian.

Other aspects of this problem were considered previously, for instance, for the case of the Einstein field equations with a cosmological constant by Wald [3], and Maeda [4] for the case of the quadratic theory $L = R+\beta R^2$ in vacuum for all Bianchi models except the IX. More specifically, Wald [3] proved, in the particular case of a true cosmological constant, that this conjecture is true for all Bianchi models, except possibly for type IX . Wald's proof served as the prototype for most subsequent works where, either a true cosmological constant was assumed [5] or the validity of the cosmic no–hair conjecture was tested in the context of particular models: chaotic inflation [6, 2], power-law inflation [7, 8], inflation due to a $R^2$ theory [4, 9, 10] or in more general $f(R)$ theories [11].

Our results contribute to an extension of previous works in two ways. We provide a detailed study of the role that matter plays in the problem and treat rigorously the

important case of the Bianchi IX model (including matter fields) *in the case of the full self-interacting potential* arising from the conformal transformation. Our proof relies on the use of an energy-like function which we call the Moss-Sahni function and is accomplished through a detailed study of energy estimates of the associated quantities that obey certain differential inequalities. We show that once the required bound for the Moss-Sahni function is obtained, the spatial part of the scalar curvature asymptotically approaches zero almost exponentially.

The field equations take the form

$$R_{ab}-\frac{1}{2}g_{ab}R-\frac{\beta}{1+2\beta R}\left(2\nabla_a\nabla_b R-2g_{ab}\Box R+\frac{1}{2}g_{ab}R^2\right)=T_{ab}\left(\mathbf{g}\right). \tag{1}$$

Under a conformal transformation of the metric

$$\widetilde{g}_{ab}=\left(1+2\beta R\right)g_{ab}, \tag{2}$$

with

$$\varphi=\sqrt{\frac{3}{2}}\ln\left(1+2\beta R\right), \tag{3}$$

the field equations (1) become the Einstein equations in the new space-time $(M,\widetilde{\mathbf{g}})$

$$\widetilde{R}_{ab}-\frac{1}{2}\widetilde{g}_{ab}\,\widetilde{R}=\nabla_a\varphi\nabla_b\varphi-\frac{1}{2}\widetilde{g}_{ab}\left(\nabla_c\varphi\nabla^c\varphi\right)-\widetilde{g}_{ab}V+\widetilde{T}_{ab}\left(\widetilde{\mathbf{g}}\right), \tag{4}$$

$$\widetilde{\Box}\varphi-V'\left(\varphi\right)=0, \tag{5}$$

where the potential $V$ is [12], [4]

$$V=\frac{1}{8\beta}\left[1-\exp\left(-\sqrt{\frac{2}{3}}\varphi\right)\right]^2. \tag{6}$$

As Maeda [4] has pointed out, this potential has a long and flat plateau. When $\varphi$ is far from the minimum of the potential, $V$ is almost constant $V_\infty\equiv\lim_{\varphi\to+\infty}V\left(\varphi\right)=1/\left(8\beta\right)$. Thus $V$ has the general properties for inflation to commence and $V_\infty$ behaves as a cosmological term. The constant $\beta$ is of order $10^{14}\ l_{PL}^2$ [13].

The plan of the paper is as follows. In the next Section we prove the cosmic no-hair conjecture for all Bianchi universes except IX, while Section 3 is devoted to the analysis of the Bianchi IX model. Precise statements of the results proved may be found in those Sections. We conclude by pointing out how our results could lead to the treatment of several more general cases.

## 2. All Bianchi models except type IX

In this Section we show using Einstein's equations (4) and the equation of motion of the scalar field (5), that, *all Bianchi models which are initially on the flat plateau of the*

*potential (6), except probably Bianchi IX, with a matter content satisfying the strong and dominant energy conditions, rapidly approach de Sitter space-time.* This is established for the full scalar field potential arising from the conformal transformation.

We shall work exclusively in the conformal picture and so for simplicity we drop the tilde. Choose $n^a$ to be the unit geodesic vector field, normal to the homogeneous hypersurfaces. The spatial metric is related to the space-time metric as $h_{ab} = g_{ab} + n_a n_b$ and $n^a \nabla_a = \partial / \partial t$, where $t$ denotes proper time along the integral curves of $n^a$. The scalar curvature $R$ becomes a function of time, hence the scalar field introduced by eq. (3) is homogeneous. Ordinary matter is assumed to satisfy the strong and dominant energy conditions, namely,

$$\begin{aligned} T_s &\equiv \left(T_{ab} - \tfrac{1}{2} T g_{ab}\right) n^a n^b \geq 0 \\ T_d &\equiv T_{ab} u^a u^b \geq 0, \quad T^a_b u^b \ \text{ non} - \text{spacelike}, \end{aligned} \tag{7}$$

for any unit timelike vector field $u^a$.

We make use of the time-time component of Einstein's equations (4)

$$\frac{1}{3} K^2 = \sigma^2 + T_d + \frac{1}{2} \dot{\varphi}^2 + V - \frac{1}{2} \, {}^{(3)}R \tag{8}$$

and the Raychaudhuri equation

$$\dot{K} = -\frac{1}{3} K^2 - 2\sigma^2 - T_s - \dot{\varphi}^2 + V. \tag{9}$$

The equation of motion for the scalar field (5) becomes

$$\ddot{\varphi} + K \dot{\varphi} + V'(\varphi) = 0. \tag{10}$$

Here $K$ is the trace of the extrinsic curvature $K_{ab}$ of the homogeneous hypersurfaces and is related to the determinant $h$ of the spatial metric by

$$K = \frac{d}{dt} \left( \ln h^{1/2} \right). \tag{11}$$

We note that in [8, 14], the system above is reduced to a two dimensional autonomous form as a consequence of the exponential form of the potential which we do not assume. Our case is more involved since the system cannot be reduced further and so we have to proceed in a different way. We make the following assumptions:

1. The initial value $\varphi_i$ of the scalar field is large and positive (i.e., the universe is on the flat plateau of the potential).
2. The kinetic energy of the field is negligible compared to the potential energy.
3. The universe is initially expanding i.e., $K > 0$ at some arbitrary time $t_i$.

With assumption (3), Eq. (8) implies that the universe will expand for all subsequent times i.e., $K > 0$ for $t \geq t_i$ for all Bianchi models, except possibly type-IX. Defining the energy density of the scalar field by $E \equiv \frac{1}{2} \dot{\varphi}^2 + V$, we find by (10) that

$$\dot{E} = -K \dot{\varphi}^2. \tag{12}$$

Hence, in an expanding universe the field looses energy and slowly rolls down the potential. The "effective" regime of inflation is the phase during the time interval $t_f - t_i$ needed for the scalar field to evolve from its initial value $\varphi_i$ to a smaller value $\varphi_f$, where $\varphi_f$ is determined by the condition that $V(\varphi_f) \simeq \eta V_\infty$. The numerical factor $\eta$ is of order say 0.9, but its precise value is irrelevant. As we shall show, the universe becomes de Sitter space during the effective regime followed by the usual FRW model when the cosmological term vanishes. To this end, we define a function $S$, which plays the same role as in Moss and Sahni [6], by

$$S = \frac{1}{3}K^2 - E. \tag{13}$$

In all Bianchi models except IX, using (8) we see that this function is non-negative due to the dominant energy condition and the fact that in these models the scalar spatial curvature is non-positive. Taking the time derivative of $S$ and using eqs. (8) and (9) we obtain

$$\dot{S} = -\frac{2}{3}KS - \frac{2}{3}\left(2\sigma^2 + T_s\right)K. \tag{14}$$

It follows that $\dot{S} \leq -(2/3)KS$ or,

$$\dot{S} \leq -\frac{2}{3}S\sqrt{3\,(S+E)}. \tag{15}$$

This differential inequality cannot be integrated immediately because $E$ is a function of time (albeit slowly-varying). However as is well known [16], in order to have inflation, $E$ must be bounded below and in this case assumption (1) above implies that we may assume that the scalar field is large enough and so, without loss of generality, we may set $E \geq \eta V_\infty$. The choice of $\eta$ affects only slightly the isotropization time, but the qualitative behavior of the model is the same. Therefore, inequality (15) implies that

$$S \leq \frac{3m^2}{\sinh^2(mt)}, \qquad m = \sqrt{\eta V_\infty/3}. \tag{16}$$

From eq. (8) we see that, as $t$ increases, the shear, three-curvature and energy density of matter rapidly approach zero and because of the dominant energy condition, all components of the energy-momentum tensor approach zero. It follows that the universe isotropizes within one Hubble time $(1/\sqrt{V_\infty} \sim 10^7\ t_{PL})$.

## 3. Bianchi type IX models

In this Section we show that *Bianchi-type IX also isotropizes if initially the scalar three-curvature* ${}^{(3)}R$ *is less than the potential* $V$ *of the scalar field.* In the present case, the scalar curvature ${}^{(3)}R$ may be positive and the argument of the previous Section does not

apply directly. In particular, the function $S$ is not bounded below from zero.† However, we can estimate an upper bound for $S$ by observing that, either $S$ is bounded above from zero, or, if $S$ is initially positive, inequality (15) implies as before that an upper bound for $S$ is given by (16). Collectively, we have the overall result that

$$S \leq \max\left\{0,\ 3m^2 \sinh^{-2}(mt)\right\}. \tag{17}$$

(Had we chosen an exponential potential, this bound would have had the form found in [8], p.1418). An estimation of a lower bound for $S$ derives from the fact that the largest positive value the spatial curvature can achieve is determined by the determinant of the three metric

$${}^{(3)}R_{\max} \propto h^{-1/3} \equiv \exp(-2\alpha). \tag{18}$$

We obtain a lower bound for $S$ in the following way. We assume that the three-scalar curvature ${}^{(3)}R$ is less than the potential energy since otherwise it is known that premature recollapse commences before vacuum domination drives the universe to inflate [17]. Then, eq. (8) immediately yields

$$\frac{1}{3}K^2 - \frac{1}{2}\dot{\varphi}^2 \geq 0.$$

Observe that for any $0 < \lambda < \sqrt{2/3}$ the above inequality implies that for $K > 0$,

$$\frac{1}{3}K - \frac{1}{2}\lambda\,\dot{\varphi} \geq \frac{1}{3}K - \frac{1}{\sqrt{6}}\left|\dot{\varphi}\right| \geq 0. \tag{19}$$

Suppose that initially i.e., at time $t_i$ we have $V > {}^{(3)}R_{\max}$. We claim that this inequality holds during the whole period of the effective regime of inflation. Following [8] we define a function $f$ by

$$f(t) \equiv \ln \frac{V}{{}^{(3)}R_{\max}}, \quad t \in [t_i, t_f], \tag{20}$$

whose initial value is positive by the above assumption. Differentiating we find

$$\dot{f} = \frac{\dot{V}}{V} + 2\,\dot{\alpha} = 2\left[\frac{\exp\left(-\sqrt{\frac{2}{3}}\varphi\right)}{1-\exp\left(-\sqrt{\frac{2}{3}}\varphi\right)}\sqrt{\frac{2}{3}}\,\dot{\varphi} + \frac{1}{3}K\right] \tag{21}$$

since $K = 3\,\dot{\alpha}$, by eq. (11). During the effective regime of inflation $V(\varphi) \geq \eta V_\infty$, hence solving for $\varphi$ this inequality we immediately verify that the coefficient of $\dot{\varphi}$ in the

† However, as Wald [3] points out, for some non-highly positively curved models premature recollapse may be avoided provided a large positive cosmological constant compensates the $-\frac{1}{2}\,{}^{(3)}R$ term in eq. (8). In our case, of course, it is the potential $V(\varphi)$ which acts as a 'cosmological term'. A similar proof to that of Wald [3] for the Bianchi IX model, was given by Kitada and Maeda [8] in the case of an exponential potential leading to power-law inflation.

brackets is less than $1/\sqrt{6}$. Therefore, inequality (19) implies that $\dot{f} \geq 0$ . We conclude that, if $K > 0$ and $f(t_i) > 0$ initially, then for $t \in [t_i, t_f]$ we obtain $f(t) \geq f(t_i)$ and our assertion follows.

We are now in a position to estimate the required bound for $S$. First, from eq. (8) it is evident that

$$-\frac{1}{2}\,{}^{(3)}R_{\max} \leq S \tag{22}$$

and since $V/2 > {}^{(3)}R_{\max}/2$, eq. (8) again implies that

$$K \geq \sqrt{\frac{3}{2}V} > \sqrt{\frac{3}{2}\eta V_\infty}. \tag{23}$$

Remembering that $K = 3\dot{\alpha}$ and using the last inequality and eq. (18), we see that ${}^{(3)}R_{\max}$ decays faster than $\exp\left[-\sqrt{2\eta V_\infty/3}\,(t-t_i)\right]$. Therefore (22) becomes

$$-\frac{1}{2}\,{}^{(3)}R_{\max}(t_i)\exp\left[-\sqrt{\frac{2}{3}\eta V_\infty}\,(t-t_i)\right] \leq S. \tag{24}$$

Combining this lower bound for $S$ with the upper bound (17), we conclude that $S$ vanishes almost exponentially. From eq. (8), $-2S \leq {}^{(3)}R \leq {}^{(3)}R_{\max}$ and therefore ${}^{(3)}R$ damps to zero just as $S$ and ${}^{(3)}R_{\max}$.

We conclude that when the universe is initially on the plateau, the shear, the scalar three-curvature and all components of the stress-energy tensor approach zero almost exponentially fast with a time constant of order $\sim 1/\sqrt{V_\infty}$.†

We discussed inflation in the equivalent space-time $(M, \widetilde{\mathbf{g}})$ , but it is not obvious that the above attractor property is maintained in the original space-time $(M, \mathbf{g})$ . This is probably an unimportant question since there is much evidence that in most relevant cases the rescaled metric $\widetilde{\mathbf{g}}$ is the real physical metric [15]. However, Maeda has pointed out [4] that inflation also naturally occurs in the original picture since during inflation the scalar field changes very slowly and the two metrics are related by $\widetilde{\mathbf{g}} = \exp\left(\sqrt{\frac{2}{3}}\varphi\right)\mathbf{g}$.

We now move on to show that *the time needed for the potential energy to reach its minimum is much larger than the time of isotropization.* Therefore, the universe reaches the potential minimum ($\varphi = 0$) at which the cosmological term vanishes and consequently evolves according to the standard Friedmann cosmology. Wald's proof assumes the existence of a cosmological constant. However, in realistic inflationary models the universe does not have a true cosmological constant but rather a vacuum energy density which during the slow evolution of the scalar field behaves like a cosmological term which eventually vaniches. Therefore we are faced with the question of whether or not the universe evolves towards a de Sitter type state *before* the potential

† Actually, the time constant for isotropization in type IX is longer by $\sqrt{2}$ than in other types. The situation is similar to that encountered in Kitada and Maeda [8].

energy of the scalar field reaches its minimum. In any consistent no-hair theorem one has to verify that the time necessary for isotropization is small compared to the time the field reaches the minimum of the potential. The following argument shows that in our case the vacuum energy is not exhausted before the universe is completely isotropized. (Similar estimates were derived in [6] for the case of the standard quadratic potential.) Imagine that at the beginning of inflation the universe is on the flat plateau of the potential. It is evident that the time $t_f - t_i$ needed for the scalar field to evolve from its initial value $\varphi_i$ to a smaller value $\varphi_f$ is smaller in the absence of damping than that in its presence. In the absence of damping, $t_f - t_i$ is easily obtained from integrating the equation of motion of the scalar field (10) ($K = 0$). Taking for example a $\varphi_f$ such that $V(\varphi_f) \simeq \eta V_\infty$ one finds that $t_f - t_i$ is more than 65 times the time $\tau$ of isotropization. The presence of damping increases the time interval $t_f - t_i$ and a larger anisotropy damps more efficiently the slow rolling of the scalar field, thus producing more inflation. It follows that when due account of damping is taken, the period of the effective regime of inflation is more than sufficient for the complete isotropization of the universe.

## 4. Conclusions

It is known [9] that the quasi-exponential solution of the $R + \beta R^2$ theory considered in this paper is an attractor to all isotropic solutions of this theory. Our analysis here suggests that the solution could be shown to attract the class of orthogonal Bianchi universes as well. If this is indeed the case, it would be interesting to ask whether this attractor is unique in the space of all higher order gravity theories. One way to directly verify this might be through a perturbation analysis and the corresponding investigation of the asymptotic structure of the solutions.

We believe that the proof of the cosmic no–hair conjecture presented here could be extended with adjustments in two directions namely, in an arbitrary number of dimensions and also to the class of titled Bianchi models. In fact, such a demonstration in some titled models would amount to a first test of this conjecture in cases of some inhomogeneity. An analysis along these lines might be more tractable than say attacking directly a genuine inhomogeneous case such as for instance that of $G_2$ cosmologies wherein the dynamics is described by systems of partial differential equations. Also a more direct analysis along the lines adopted here could lead to an extension of our results in higher dimensions.

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